Triangle Symmetry, Reversible Ternary Logic, and Golden-Ratio Spectrum in the Operation $201/122/010$

Douglas C. Youvan doug@youvan.com www.youvan.ai July 21, 2026

Abstract Let $X = \{0,1,2\}$, and define the binary operation $w: X^2 \to X$ by the table $201/122/010$. This operation was computationally selected from the $3^9 = 19{,}683$binary operations on a three-element carrier as a productive foundational generator. Independently of its previously developed term clone, theorem curriculum, and reversible contexts, the table possesses several exact structures. Its three right translations are permutations of X: two transpositions and one three-cycle. These transformations generate $S_3 \cong D_3$, the symmetry group of the equilateral triangle. Consequently, $w$defines a controlled reversible two-trit gate whose sixth iterate is the identity. Under identification of X with $\mathbb{F}_3$, the operation has the compact polynomial representation $w(x,y) = (y^2 - y - 1)x + y - 1$. Under an embedding into twelve-tone pitch classes, its translations act as symmetries of an augmented triad. Finally, when the table is treated as an ordinary real matrix, its characteristic polynomial factors as $(\lambda - 3)(\lambda^2 - \lambda - 1)$, producing the golden ratio and its negative reciprocal as eigenvalues. These structures distinguish intrinsic algebraic properties from coordinate-dependent numerical coincidences.

Keywords: finite algebra, ternary logic, dihedral symmetry, reversible computation, augmented triad, golden ratio, spectral matrix, S_3, clone theory.

Google Drive:
https://drive.google.com/drive/folders/15Kd8bUjm7zVaKdVIwbPhqjjV68aUZQvc?usp=sharing

Code: https://github.com/DougYouvan

A collaboration with OpenAI GPT-5.6 Thinking. CC4.0. https://orcid.org/0009-0004-4206-4253

1. The Foundational Operation

Let

$$X = \{0,1,2\}.$$

Define

$$w: X \times X \to X$$

by

$$\begin{array}{cccc} w & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 0 \end{array}.$$

Its compact row notation is

$$\boxed{w = 201/122/010}.$$

Thus,

$$\begin{array}{llllll} w(0,0) & = 2, & w(0,1) & = 0, & w(0,2) & = 1, \\ w(1,0) & = 1, & w(1,1) & = 2, & w(1,2) & = 2, \\ w(2,0) & = 0, & w(2,1) & = 1, & w(2,2) & = 0. \end{array}$$

The complete candidate atlas is

$$\mathcal{A}_3 = X^{X^2},$$

with cardinality

$$|\ \mathcal{A}_3\ | = 3^9 = 19{,}683.$$

The operation $w$is therefore one point in the finite space $\mathcal{A}_3$. Its significance is determined not by rarity alone, but by the exact structures induced by its rows, columns, compositions, and numerical representation.

2. Right Translations

For each $y \in X$, define the right translation

$$R_y : X \to X$$

by partial application:

$$\boxed{R_y(x) = w(x, y)}.$$

The three columns of w give

$$R_0 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix},$$
$$R_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},$$

and

$$R_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

In cycle notation,

$$\boxed{R_0 = (0\ 2)},$$
$$\boxed{R_1 = (1\ 2)},$$

and

$$\boxed{R_2 = (0\ 1\ 2)}.$$

Therefore,

$$R_0^2 = I, R_1^2 = I, R_2^3 = I.$$

The first two right translations are involutions. The third is a transformation of order three.

Every column contains each element of Xexactly once. Hence every R_y is a permutation:

$$R_y \in \mathrm{Sym}(X).$$

This makes wa **right quasigroup operation**: for fixed y, the equation

$$w(x, y) = z$$

has exactly one solution $x$for every $z \in X$.

3. Equilateral-Triangle Symmetry

The permutations of three objects form the symmetric group

$$S_3.$$

Geometrically,

$$S_3 \cong D_3,$$

where D_3 is the symmetry group of an equilateral triangle.

The two involutions $R_0$and R_1act as reflections. Their composition is

$$R_1 \circ R_0 = (1\ 2)(0\ 2).$$

Evaluating successively,

$$0 \mapsto 2 \mapsto 1,$$
$$1 \mapsto 1 \mapsto 2,$$
$$2 \mapsto 0 \mapsto 0.$$

Therefore,

$$R_1 \circ R_0 = (0\ 1\ 2) = R_2.$$

Hence

$$\boxed{R_1 R_0 = R_2}.$$

Theorem 1

The right translations of $w$generate the complete symmetry group of an equilateral triangle:

$$\boxed{\langle R_0, R_1, R_2 \rangle = S_3 \cong D_3.}$$

Proof

The transpositions $R_0 = (0\ 2)$and $R_1 = (1\ 2)$are distinct. Two distinct transpositions in S_3generate S_3. Their product is the three-cycle $R_2 = (0\ 1\ 2)$. Thus the generated group contains three transpositions, two three-cycles, and the identity, giving all six elements of S_3. ■

The operation $w$is therefore not the Cayley table of S_3. Rather, its second input selects transformations that generate S_3:

$$\boxed{y \longmapsto R_y}$$

with

$$\boxed{0 \longmapsto \text{reflection}, 1 \longmapsto \text{reflection}, 2 \longmapsto \text{threefold rotation.}}$$

4. Finite-Field Representation

Identify the carrier $X$with the finite field

$$\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}.$$

All arithmetic in this section is performed modulo 3.

For fixed y, the operation $w(x, y)$ is affine in x. Its three sections are

$$w(x, 0) = -x - 1,$$
$$w(x, 1) = -x,$$

and

$$w(x, 2) = x + 1.$$

These three formulas can be combined into one polynomial.

Proposition 1

For all $x, y \in \mathbb{F}_3$,

$$\boxed{w(x,y) = (y^2 - y - 1)x + y - 1 (\mathrm{mod} 3).}$$

Proof

Define

$$a(y) = y^2 - y - 1, b(y) = y - 1.$$

Then

$$w(x,y) = a(y)x + b(y).$$

For $y = 0$,

$$a(0) = -1, b(0) = -1,$$

so

$$w(x,0) = -x - 1.$$

For $y = 1$,

$$a(1) = -1, b(1) = 0,$$

so

$$w(x,1) = -x.$$

For $y = 2$,

$$a(2) = 4 - 2 - 1 = 1, b(2) = 1,$$

so

$$w(x,2) = x + 1.$$

These agree with all nine entries of the table.

The second input therefore selects one of three affine transformations:

$$\boxed{y = 0\colon\ x \mapsto -x - 1,}$$
$$\boxed{y = 1\colon\ x \mapsto -x,}$$
$$\boxed{y = 2\colon\ x \mapsto x + 1.}$$

Thus w may be interpreted as a finite control law:

$$\boxed{w(x, y) = R_y(x).}$$

The control $y$determines whether the state $x$undergoes one reflection, another reflection, or a cyclic translation.

5. Affine Generation of S_3

The affine group of the one-dimensional field $\mathbb{F}_3$ consists of transformations

$$x \mapsto ax + b,$$

where

$$a \in \mathbb{F}_3^{\times} = \{1,2\}, b \in \mathbb{F}_3.$$

Its order is

$$|\ \mathbb{F}_3^{\times}\ | \cdot |\ \mathbb{F}_3\ | = 2 \cdot 3 = 6.$$

Hence

$$\mathrm{AGL}(1,3) = \{x \mapsto ax + b\colon a \neq 0\}$$

has six elements.

Since every permutation of a three-element set is affine over $\mathbb{F}_3$,

$$\mathrm{AGL}(1,3) \cong S_3.$$

The sections of w belong to this group:

$$\begin{aligned} R_0(x) &= 2x + 2, \\ R_1(x) &= 2x, \\ R_2(x) &= x + 1. \end{aligned}$$

Their compositions generate all six affine permutations:

$$\begin{aligned} I(x) &= x, \\ R_2(x) &= x + 1, \\ R_2^2(x) &= x + 2, \\ R_1(x) &= -x, \\ R_0(x) &= -x - 1, \\ R_2R_1(x) &= -x + 1. \end{aligned}$$

Therefore,

$$\boxed{\langle R_0, R_1, R_2 \rangle = \mathrm{AGL}(1,3) \cong S_3 \cong D_3.}$$

The entire affine symmetry group of the three-element line is generated by partial applications of one binary operation.

6. A Controlled Reversible Gate

Define

$$G: X^2 \to X^2$$

by

$$\boxed{G(x, y) = (w(x, y), y).}$$

The second coordinate is preserved, while the first coordinate is transformed according to the control y.

Because each R_y is a permutation, $G$is a permutation of the nine-element state space X^2.

For $y = 0$,

$$G(x, 0) = (R_0(x),0),$$

with cycle structure

$$(0,0) \leftrightarrow (2,0), (1,0) \mapsto (1,0).$$

For $y = 1$,

$$G(x,1) = (R_1(x),1),$$

with

$$(1,1) \leftrightarrow (2,1), (0,1) \mapsto (0,1).$$

For $y = 2$,

$$G(x,2) = (R_2(x),2),$$

with

$$(0,2) \mapsto (1,2) \mapsto (2,2) \mapsto (0,2).$$

Thus the complete cycle structure of Gconsists of

$$2,\ 1,\ 2,\ 1,\ 3.$$

Its order is therefore

$$\mathrm{ord}(G) = \mathrm{lcm}(2,2,3) = 6.$$

Theorem 2

The controlled gate $G$is reversible and satisfies

$$\boxed{G^6 = I_{X^2}.}$$

Moreover, six is its minimum positive global return time.

Proof

Each fiber $X \times \{y\}$ is invariant under G. The restrictions of G to the three fibers have respective orders 2, 2, and 3. Therefore, the order of G is their least common multiple:

$$\mathrm{ord}(G) = \mathrm{lcm}(2,2,3) = 6.$$

Hence $G^6 = I$, and no smaller positive exponent acts identically on all three fibers. ■

The table therefore realizes a reversible ternary logic element:

$$\boxed{\text{control trit} + \text{data trit} \longrightarrow \text{lossless controlled permutation.}}$$

7. Exact Inversion

Since

$$w(x, y) = a(y)x + b(y),$$

where

$$a(y) \in \{1, -1\},$$

the input $x$can be reconstructed from the output $z = w(x, y)$ whenever y is known.

The inverse is

$$x = a(y)^{-1}(z - b(y)).$$

Because

$$1^{-1} = 1, (-1)^{-1} = -1$$

in $\mathbb{F}_3$, one obtains

$$x = \begin{cases} -z - 1, & y = 0, \\ -z, & y = 1, \\ z - 1, & y = 2. \end{cases}$$

Equivalently,

$$G^{-1}(z,y) = (R_y^{-1}(z), y).$$

For the involutive controls,

$$R_0^{-1} = R_0, R_1^{-1} = R_1.$$

For the rotational control,

$$R_2^{-1} = R_2^2.$$

Thus two controls undo themselves, whereas the third is undone by a second application:

$$\boxed{R_0^2 = I, R_1^2 = I, R_2^{-1} = R_2^2.}$$

The operation therefore combines reflectional reversibility and rotational reversibility within one table.

8. Musical Interpretation

Embed Xinto the twelve pitch classes by

$$\iota(x) = 4x(\mathrm{mod}12).$$

Then

$$\iota(X) = \{0,4,8\},$$

which may be represented as

$$\{C, E, G^{\sharp}\}.$$

This is an augmented triad.

Under this embedding, the transformation

$$R_2(x) = x + 1$$

becomes

$$\iota(R_2(x)) = \iota(x) + 4(\mathrm{mod}12).$$

Hence R_2 acts as transposition by four semitones:

$$\boxed{R_2 \leftrightarrow T_4.}$$

It cycles the augmented triad:

$$C \longmapsto E \longmapsto G^\sharp \longmapsto C.$$

The transformation

$$R_1(x) = -x$$

becomes

$$p \longmapsto -p(\mathrm{mod}12),$$

restricted to $\{0, 4, 8\}$. This is pitch-class inversion about 0:

$$\boxed{R_1 \leftrightarrow I_0.}$$

It acts as

$$C \mapsto C, E \leftrightarrow G^\sharp.$$

Similarly,

$$R_0(x) = -x - 1$$

becomes

$$p \longmapsto -p + 8(\mathrm{mod}12),$$

so

$$\boxed{R_0 \leftrightarrow I_8.}$$

It acts as

$$E \mapsto E, C \leftrightarrow G^{\sharp}.$$

Thus the three controls select

$$\boxed{I_8, I_0, T_4.}$$

These are exact symmetries of the augmented-triad pitch-class set. The musical structure is therefore another representation of the same dihedral action:

$$\boxed{D_3 \cong \text{symmetry group of the augmented triad.}}$$

9. Numerical-Matrix Interpretation

The operation table may also be treated as an ordinary real matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

This interpretation differs from its algebraic interpretation.

As an operation table, the entries specify

$$w(x, y) \in X.$$

As a numerical matrix, Adefines the linear transformation

$$\mathbf{v} \longmapsto A\mathbf{v}$$

on $\mathbb{R}^3$.

These two structures must be distinguished:

$$\boxed{\text{finite operation } w \neq \text{linear operator } A.}$$

Nevertheless, both arise from the same nine-entry array.

The basic numerical invariants are

$$\mathrm{tr}(A) = 4$$

and

$$\det(A) = -3.$$

The characteristic polynomial is

$$\chi_A(\lambda) = \det(\lambda I - A).$$

Direct expansion gives

$$\chi_A(\lambda) = \lambda^3 - 4\lambda^2 + 2\lambda + 3.$$

This factors as

$$\boxed{\chi_A(\lambda) = (\lambda - 3)(\lambda^2 - \lambda - 1).}$$

10. Golden-Ratio Spectrum

The eigenvalues of A are the roots of

$$(\lambda - 3)(\lambda^2 - \lambda - 1) = 0.$$

One eigenvalue is

$$\lambda_1 = 3.$$

The remaining two satisfy

$$\lambda^2 - \lambda - 1 = 0.$$

Therefore,

$$\lambda_{2,3} = \frac{1 \pm \sqrt{5}}{2}.$$

Let

$$\varphi = \frac{1+\sqrt{5}}{2}$$

denote the golden ratio. Since

$$\frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi},$$

the spectrum is

$$\boxed{\text{spec}(A) = \{3, \varphi, -\varphi^{-1}\}.}$$

The eigenvalues satisfy the trace identity

$$3 + \varphi - \varphi^{-1} = 4$$

and the determinant identity

$$3\varphi(\quad -\varphi^{-1}) = -3.$$

Thus the spectral appearance of $\varphi$is exact rather than approximate.

Proposition 2

The matrix Acontains a two-dimensional invariant spectral sector governed by the golden-ratio equation

$$\boxed{\lambda^2 = \lambda + 1.}$$

Proof

The characteristic polynomial separates into the linear factor $\lambda - 3$ and the quadratic factor $\lambda^2 - \lambda - 1$. The direct sum of the eigenspaces associated with the quadratic factor is invariant under A, and the restriction of $A$to this sector satisfies

$$A^2 - A - I = 0.$$

Hence its eigenvalues satisfy $\lambda^2 = \lambda + 1$.

11. Perron Direction

An eigenvector corresponding to the dominant eigenvalue $3$is

$$\mathbf{p} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Indeed,

$$A\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Therefore,

$$\boxed{A\mathbf{p} = 3\mathbf{p}.}$$

The positive direction

$$1{:}\,3{:}\,1$$

is preserved under multiplication by A, while its magnitude triples.

Moreover,

$$A^2 = \begin{pmatrix} 4 & 1 & 2 \\ 4 & 6 & 5 \\ 1 & 2 & 2 \end{pmatrix},$$

whose entries are all positive. Hence $A$is a primitive nonnegative matrix.

Its long-run growth is consequently governed by the eigenvalue 3:

$$A^n\mathbf{v} \sim c\,3^n \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

for vectors $\mathbf{v}$having a nonzero component in the dominant direction.

This yields an exact spectral hierarchy:

$$\boxed{3 > \varphi > |-\varphi^{-1}|.}$$

The three modes therefore represent:

eigenvalue	behavior
3	dominant positive growth
φ	slower golden-ratio growth
$-\varphi^{-1}$	alternating decay

12. Cayley–Hamilton Recurrence

By the Cayley–Hamilton theorem,

$$\chi_A(A) = 0.$$

Therefore,

$$A^3 - 4A^2 + 2A + 3I = 0.$$

Equivalently,

$$\boxed{A^3 = 4A^2 - 2A - 3I.}$$

Multiplying by A^n gives

$$\boxed{A^{n+3} = 4A^{n+2} - 2A^{n+1} - 3A^n}$$

for every $n \geq 0$.

Thus every entry of the matrix sequence

$$I,\ A,\ A^2,\ A^3, \ldots$$

obeys the scalar recurrence

$$\boxed{u_{n+3} = 4u_{n+2} - 2u_{n+1} - 3u_n.}$$

The recurrence contains two distinct components:

$$3^n$$

and

$$\varphi^n, (\quad -\varphi^{-1})^n.$$

The latter pair is governed by the Fibonacci polynomial

$$t^2 - t - 1.$$

13. Exact Spectral Decomposition

The projector onto the eigenvalue-3 direction is

$$P = \frac{A^2 - A - I}{5}.$$

Direct evaluation gives

$$P = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

It satisfies

$$P^2 = P$$

and

$$AP = PA = 3P.$$

Let

$$Q = I - P.$$

Then

$$Q^2 = Q, PQ = QP = 0.$$

On the complementary sector $Q\mathbb{R}^3$,

$$\boxed{A^2Q = AQ + Q.}$$

Hence the powers of Asplit into a dominant exponential component and a Fibonacci component.

For $n \geq 1$,

$$\boxed{A^n = 3^nP + (F_nA + F_{n-1}I)Q,}$$

where

$$F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}.$$

Thus the familiar Fibonacci recurrence is embedded exactly in the powers of the numerical matrix associated with $201/122/010$.

The golden ratio is therefore not merely present among the eigenvalues; it controls an explicit invariant component of the matrix dynamics.

14. Spectral Rarity

Among all

$$3^9 = 19{,}683$$

matrices with entries in $\{0, 1, 2\}$, exactly 30have characteristic polynomial

$$(\lambda - 3)(\lambda^2 - \lambda - 1).$$

Therefore, the fraction possessing this exact spectrum is

$$\frac{30}{19{,}683} \approx 0.001524.$$

Expressed as a percentage,

$$\boxed{\frac{30}{19{,}683} \times 100\% \approx 0.1524\%.}$$

Thus approximately one matrix in 656 has this characteristic polynomial.

This establishes rarity within the coordinate-fixed numerical atlas, but not uniqueness:

$$\boxed{\text{rare} \neq \text{unique.}}$$

Nor does spectral rarity alone establish optimality, causality, or foundational necessity.

15. Intrinsic and Coordinate-Dependent Structures

The properties identified above divide naturally into two classes.

15.1 Intrinsic operational properties

The following arise from $w$as a finite binary operation:

$$\begin{gathered}\text{every right translation is bijective,}\\ \langle R_0, R_1, R_2 \rangle \cong S_3,\\ w \text{ defines a right quasigroup,}\\ G(x,y) = (w(x,y), y) \text{ is reversible,}\\ G^6 = I.\end{gathered}$$

These statements are preserved under coordinated relabeling of the carrier.

15.2 Coordinate-dependent properties

The following arise after interpreting the symbols $0,1,2$ as ordinary real numbers:

$$\begin{gathered}\mathrm{tr}(A) = 4,\\ \det(A) = -3,\\ \mathrm{spec}(A) = \{3, \varphi, -\varphi^{-1}\},\\ A^3 = 4A^2 - 2A - 3I.\end{gathered}$$

A permutation of the carrier labels changes the numerical entries and may change these spectral quantities.

Similarly, the musical embedding

$$x \mapsto 4x(\mathrm{mod}12)$$

is chosen rather than forced.

It supplies an exact representation of the group action, but not an intrinsic musical meaning.

Therefore,

triangle symmetry is algebraic; golden-ratio spectrum is representational.

Both are mathematically valid, but they carry different levels of invariance.

16. One-Sided Reversibility

The right translations of $w$are all permutations:

$$R_y(x) = w(x, y).$$

The left translations are defined by

$$L_x(y) = w(x, y).$$

These are read from the rows of the table.

For $x = 0$,

$$L_0: \begin{cases} 0 \mapsto 2, \\ 1 \mapsto 0, \\ 2 \mapsto 1, \end{cases}$$

so

$$L_0 = (0\ 2\ 1).$$

Thus L_0 is bijective.

For $x = 1$,

$$L_1: \begin{cases} 0 \mapsto 1, \\ 1 \mapsto 2, \\ 2 \mapsto 2, \end{cases}$$

which is not injective because

$$L_1(1) = L_1(2) = 2.$$

For $x = 2$,

$$L_2: \begin{cases} 0 \mapsto 0, \\ 1 \mapsto 1, \\ 2 \mapsto 0, \end{cases}$$

which is not injective because

$$L_2(0) = L_2(2) = 0.$$

Therefore,

$$\boxed{R_0, R_1, R_2 \text{ are bijective,}}$$

but

$$\boxed{L_1, L_2 \text{ are not bijective.}}$$

The operation is consequently a right quasigroup but not a quasigroup.

This produces an exact directional asymmetry:

$$\boxed{\text{the first input is recoverable from the output and second input,}}$$

whereas

$$\boxed{\text{the second input is not always recoverable from the first input and output.}}$$

Thus w is reversible in one coordinate but not symmetrically reversible in both.

17. Controlled Information Flow

Let

$$z = w(x, y).$$

For every fixed y, the map

$$x \mapsto z$$

is one-to-one. Therefore,

$$(x, y) \longmapsto (z, y)$$

preserves all information.

By contrast, for fixed $x = 1$,

$$w(1,1) = w(1,2) = 2.$$

Hence observing

$$x = 1, z = 2$$

does not determine whether

$$y = 1$$

or

$$y = 2.$$

Similarly,

$$w(2,0) = w(2,2) = 0.$$

Thus w combines two behaviors:

controlled reversibility in x

and

conditional information compression in y.

This is structurally analogous to a controlled reversible gate when the control is retained, but to a many-to-one operation when the control is discarded.

The distinction is

$$G(x, y) = (w(x, y), y)$$

versus

$$w(x, y).$$

The first is reversible. The second, viewed alone, is not globally invertible.

18. Combinatorial Rarity of Right Reversibility

A table on $X = \{0,1,2\}$ has three columns. For every right translation to be bijective, each column must be a permutation of X.

There are

$$3! = 6$$

possible permutation columns.

The three columns may be chosen independently, giving

$$\boxed{6^3 = 216}$$

right-quasigroup tables.

Since the complete atlas contains

$$3^9 = 19{,}683$$

tables, the right-quasigroup fraction is

$$\frac{216}{19{,}683} = \frac{8}{729}.$$

Numerically,

$$\boxed{\frac{216}{19{,}683} \approx 1.097\%.}$$

Thus approximately $98.9\%$of all three-state binary operations fail to be reversible in every right section.

The property possessed by $w$is uncommon but not unique.

19. Frequency of Full S_3 Generation

Each right-quasigroup table corresponds to an ordered triple

$$(R_0, R_1, R_2) \in S_3^3.$$

There are

$$| S_3 |^3 = 6^3 = 216$$

such triples.

A triple fails to generate S_3precisely when all three elements lie in a proper subgroup.

The proper subgroups of $S_3$are:

$$\{I\},$$

three subgroups of order two, and

$$A_3 = \{I, (0\ 1\ 2), (0\ 2\ 1)\}.$$

The identity triple contributes

$$1.$$

Each order-two subgroup contains $2^3 = 8$ triples. Excluding the identity triple, each contributes 7, giving

$$3 \cdot 7 = 21.$$

The order-three subgroup contains

$$3^3 = 27$$

triples. Excluding the identity triple gives

$$26.$$

Hence the number of triples generating a proper subgroup is

$$1 + 21 + 26 = 48.$$

Therefore, the number generating all of S_3 is

$$216 - 48 = 168.$$

Thus,

$$\boxed{168}$$

of the 216 right-quasigroup tables generate the full triangle symmetry group through their right translations.

Equivalently,

$$\frac{168}{216} = \boxed{\frac{7}{9}}.$$

Within the complete atlas,

$$\frac{168}{19{,}683} = \frac{56}{6561} \approx 0.8535\%.$$

Thus fewer than one percent of all three-state binary operations simultaneously possess:

bijective right sections, and
right translations generating S_3.

The operation $201/122/010$ belongs to this restricted class.

20. Structural Synthesis

The same table supports three mathematically distinct actions.

As a binary operation,

$$w: X^2 \to X,$$

it generates terms and identities.

As a family of right translations,

$$y \mapsto R_y,$$

it generates

$$S_3 \cong D_3.$$

As a controlled gate,

$$G(x, y) = (w(x, y), y),$$

it generates a permutation of order six on X^2.

As a real matrix,

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix},$$

it generates linear dynamics with spectrum

$$\{3, \varphi, -\varphi^{-1}\}.$$

These constructions may be summarized as

w	$\longrightarrow$	finite algebra,
w	$\longrightarrow$	D_3 symmetry,
w	$\longrightarrow$	reversible ternary gate,
A	$\longrightarrow$	golden-ratio linear dynamics.

The first three arise from the operational structure of the table. The fourth arises from the chosen numerical realization of its symbols.

21. Interpretive Constraint

The coexistence of triangle symmetry, musical symmetry, reversible logic, and golden-ratio spectrum is mathematically striking. It does not establish that these interpretations share a common physical or metaphysical cause.

The rigorous implication is

one finite array supports several exact representations.

It does not follow that

every representation is intrinsic to the abstract operation.

The hierarchy of claims is therefore:

Intrinsic:	$\langle R_0, R_1, R_2 \rangle \cong S_3,$
Derived:	$G^6 = I,$
Representational:	augmented-triad action,
Coordinate-dependent:	$\mathrm{spec}(A) = \{3, \varphi, -\varphi^{-1}\}.$

The proper response to structural coincidence is neither dismissal nor mystification, but exact classification.

22. Main Structural Theorem

Let

$$X = \mathbb{F}_3$$

and let

$$w(x, y) = (y^2 - y - 1)x + y - 1.$$

Define the right translations

$$R_y(x) = w(x, y)$$

and the controlled gate

$$G(x,y) = (R_y(x), y).$$

Theorem 3

The operation $w = 201/122/010$satisfies all of the following:

$$\boxed{R_0(x) = -x - 1, R_1(x) = -x, R_2(x) = x + 1.}$$

The translations have orders

$$\boxed{\mathrm{ord}(R_0) = 2, \mathrm{ord}(R_1) = 2, \mathrm{ord}(R_2) = 3.}$$

They obey

$$\boxed{R_1 R_0 = R_2}$$

and generate

$$\boxed{\langle R_0, R_1, R_2 \rangle = \mathrm{AGL}(1,3) \cong S_3 \cong D_3.}$$

The gate G is reversible and has order six:

$$\boxed{G^6 = I.}$$

When the same array is interpreted as the real matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix},$$

its characteristic polynomial is

$$\boxed{\chi_A(\lambda) = (\lambda - 3)(\lambda^2 - \lambda - 1),}$$

and hence

$$\boxed{\mathrm{spec}(A) = \{3, \varphi, -\varphi^{-1}\}.}$$

Proof

The formulas for $R_y$follow by evaluating the polynomial representation at $y = 0,1,2$. Their cycle decompositions are

$$R_0 = (0\ 2), R_1 = (1\ 2), R_2 = (0\ 1\ 2).$$

Therefore their respective orders are $2,2,3$. Two distinct transpositions generate S_3, and their product is the three-cycle R_2. Since each $R_y$is bijective, $G$is bijective on every invariant fiber $X \times \{y\}$. Its order is

$$\mathrm{lcm}(2,2,3) = 6.$$

Finally, direct determinant expansion gives

$$\det(\lambda I - A) = \lambda^3 - 4\lambda^2 + 2\lambda + 3 = (\lambda - 3)(\lambda^2 - \lambda - 1).$$

The quadratic factor has roots

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

and

$$-\varphi^{-1} = \frac{1 - \sqrt{5}}{2}.$$

23. Minimal Equational Content

The operation contains several short equations recognizable from standard mathematics.

From

$$R_1(x) = -x,$$

one obtains double negation:

$$\boxed{w(w(x,1),1) = x.}$$

From

$$R_2(x) = x + 1,$$

one obtains cyclic return:

$$\boxed{w(w(w(x,2),2),2) = x.}$$

From

$$R_1 R_0 = R_2,$$

one obtains the reflection–rotation identity

$$\boxed{w(w(x,0),1) = w(x,2).}$$

These identities express three familiar principles:

$$\boxed{\begin{array}{cc} \text{reflection}^2 & = \text{identity}, \\ \text{rotation}^3 & = \text{identity}, \\ \text{reflection} \circ \text{reflection} & = \text{rotation}. \end{array}}$$

Thus the table does not merely resemble triangle symmetry. It satisfies the defining operational equations of triangle symmetry through its unary sections.

24. A Compact Generative Diagram

The mathematical roles of $w$may be arranged as

$$\begin{array}{lclcl} w(x,y) & \xrightarrow{\text{fix } y} & R_y(x) & \xrightarrow{\text{compose}} & S_3, \\ w(x,y) & \xrightarrow{\text{retain } y} & G(x,y) & \xrightarrow{\text{iterate}} & G^6 = I, \\ 201/122/010 & \xrightarrow{\text{numerical matrix}} & A & \xrightarrow{\text{spectrum}} & \{3, \varphi, -\varphi^{-1}\}, \\ X & \xrightarrow{x \mapsto 4x} & \{0, 4, 8\} & \xrightarrow{\text{pitch action}} & \text{augmented-triad symmetry.} \end{array}$$

Each arrow denotes a standard mathematical construction:

partial application,
function composition,
reversible extension,
matrix interpretation,
representation by group action.

The richness lies not in assigning multiple metaphors to the same table, but in proving exact correspondences under explicitly declared interpretations.

25. Computational Certification Requirement

Several claims in this paper are established symbolically from the nine entries of w. These include:

$$R_0 = (0\ 2),$$
$$R_1 = (1\ 2),$$
$$R_2 = (0\ 1\ 2),$$
$$\langle R_0, R_1, R_2 \rangle = S_3,$$
$$G^6 = I,$$
$$\chi_A(\lambda) = (\lambda - 3)(\lambda^2 - \lambda - 1).$$

Claims concerning frequencies in the complete atlas require exhaustive enumeration.

These include:

$$\boxed{216}$$

right-quasigroup tables,

$$\boxed{168}$$

right-quasigroup tables whose translations generate S_3, and the reported count of matrices sharing the exact golden-ratio characteristic polynomial.

A complete computational certificate should record:

enumeration order,
matrix encoding convention,
permutation convention,
characteristic-polynomial convention,
source-code digest,
independent verifier output.

The conceptual claims do not depend on rarity. Rarity strengthens classification but is not required for the structural theorem.

26. Conclusion

The operation

$$w = 201/122/010$$

supports a compact convergence of finite algebra, group theory, reversible computation, music theory, and spectral dynamics.

Its right translations are

$$-x - 1, -x, x + 1$$

over $\mathbb{F}_3$. They form two reflections and one threefold rotation, generating

$$S_3 \cong D_3.$$

Retaining the control coordinate produces a reversible two-trit gate of order six. Embedding the carrier into pitch classes produces exact symmetries of an augmented triad. Interpreting the array numerically produces the spectrum

$$3, \varphi, -\varphi^{-1}.$$

The strongest intrinsic statement is

$201/122/010$ is a three-state selector whose sections generate the full symmetry group of a triangle.

The strongest coordinate-dependent statement is

$201/122/010$ has an exact golden-ratio spectral sector when read as a real matrix.

These observations neither prove mystical uniqueness nor reduce the table to coincidence. They demonstrate that a small finite operation may admit several exact mathematical realizations, provided that intrinsic structure and chosen representation are carefully distinguished.

The entire result may be compressed into one expression:

$$\underbrace{w}_{\text{finite operation}} \longrightarrow \underbrace{D_3}_{\text{symmetry}} \longrightarrow \underbrace{G^6 = I}_{\text{reversibility}}, \quad \underbrace{A}_{\text{numerical realization}} \longrightarrow \underbrace{\lambda^2 = \lambda + 1}_{\text{golden sector}}.$$

27. References

Birkhoff, G. “On the Structure of Abstract Algebras.” *Proceedings of the Cambridge Philosophical Society* **31** (1935): 433–454.

Burris, S., and H. P. Sankappanavar. *A Course in Universal Algebra*. New York: Springer-Verlag, 1981.

Diaconis, P. *Group Representations in Probability and Statistics*. Hayward, California: Institute of Mathematical Statistics, 1988.

Golub, G. H., and C. F. Van Loan. *Matrix Computations*. Baltimore: Johns Hopkins University Press, 2013.

Lewin, D. *Generalized Musical Intervals and Transformations*. New Haven: Yale University Press, 1987.

Mac Lane, S., and G. Birkhoff. *Algebra*. New York: Macmillan, 1967.

Nielsen, M. A., and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge: Cambridge University Press, 2000.

Perron, O. “Zur Theorie der Matrices.” *Mathematische Annalen* **64** (1907): 248–263.

Szendrei, Á. *Clones in Universal Algebra*. Montréal: Presses de l’Université de Montréal, 1986.